

Canonical LQG operators and kinematical states for plane gravitational waves

F. Hinterleitner,
Department of theoretical physics and astrophysics,
Masaryk university Brno, Czech Republic

March 13, 2017

Abstract

In a 1+1 dimensional model of plane gravitational waves the flux-holonomy algebra of loop quantum gravity is modified in such a way that the new basic operators satisfy canonical commutation relations. Thanks to this construction it is possible to find kinematical solutions for unidirectional plane gravitational waves with finite geometric expectation values and fluctuations, which was problematic in a more conventional approach in a foregoing paper by the author and coauthors [1].

1 Introduction

Nonperturbative canonical quantum gravity comes in two steps: The first one is a formulation of general relativity in terms of connection and triads on a spacelike hypersurface - the Ashtekar variables - where the total Hamiltonian is a combination of constraints. The constraints form a first-class Poisson bracket algebra.

In the second step quantum operators and states are constructed. In this process, not the connection components themselves, but their holonomies play the role of fundamental variables. Thus, before promoting the constraints or other functions of the connection to operators, the connection has to be reformulated in terms of holonomies, in such a way that for weak gravitational fields and in the continuous limit the original formulations are approximated. This leads to the problem that the Poisson bracket algebra of constraints does not carry over identically to the commutator algebra of the corresponding constraint operators. The present approach to a simplified 1+1 dimensional model is guided by two principles:

1) We construct slightly modified operators following the prototypes in elementary quantum mechanics, with configuration variables promoted to multiplication operators and conjugate momenta to derivatives. In loop quantum gravity (LQG) state functions are functions of group elements (holonomies), so we introduce in section 3 as fundamental operators multiplication by group elements in the fundamental representation and derivative operators with respect to them, instead of derivatives with respect to Lie algebra elements. In this way the fundamental operators commute canonically.

2) In LQG eigenvalues of triad operators usually have both signs, which leads, in contrast to classical theory, to identical copies of the metric geometry with different orientations of spatial directions. It is natural that quantum operators, as far as they are not related to spatial orientation, should act in an equivalent way in sectors of geometry differing only by orientation. This leads to slightly different, but quite natural constructions of corresponding operators in different sectors. In the calculations in section 4 it turns out that such a choice is necessary for physically acceptable results in all sectors.

The model, which our attention is directed to in this paper, is a model of plane gravitational waves [2], derived from a Gowdy model formulated in Ashtekar variables in [3, 4]. Being homogeneous in two directions, this is an example of an effectively 1+1 dimensional midi-superspace. In this model the new construction of canonically commuting operators is applied to the formulation of a unidirectionality constraint operator and its solutions. Another interesting approach to models of this type with a modification of operators is the abelianization of the Hamiltonian constraint [5],

2 The model

In the model of plane gravitational waves, the physical object of this paper, we assume homogeneity in the (x, y) plane and propagation in the z direction. As a further simplification we assume linearly polarized waves. The Ashtekar variables are the following: Connection components X , Y in the x and y direction and \mathcal{A} in the z direction, and respective conjugate densitized triads E^x , E^y , and \mathcal{E} . On a spacelike hypersurface all these variables depend only on z . In terms of these variables the spatial metric has the form

$$ds^2 = \mathcal{E} \frac{E^y}{E^x} dx^2 + \mathcal{E} \frac{E^x}{E^y} dy^2 + \frac{E^x E^y}{\mathcal{E}} dz^2. \quad (1)$$

The Gauß, diffeo, and Hamiltonian constraint of the system are given in [3, 4].

The graph G , on which one-dimensional analogs of spin networks (SNW) are defined, is the z axis, divided into a sequence of links l_i by nodes n_i at the locations z_i . In [4] basic quantum state functions are constructed from the point holonomies $\exp(i\frac{\mu_i}{2} X(z_i))$ and $\exp(i\frac{\nu_i}{2} Y(z_i))$ at the nodes and the link holonomies $\exp(i\frac{k_i}{2} \int_{l_i} \mathcal{A})$. The point holonomies lie in \mathbf{R}_{Bohr} , the group

of the Bohr compactification of the reals, link holonomies are $U(1)$ functions. The combined state functions (for convenience without the factors of $1/2$ in the exponents present in [4]) are

$$\prod_{\ell_j \in G} \exp \left(i k_j \int_{\ell_j} \mathcal{A} \right) \prod_{n_i \in N(G)} \exp(i \mu_i X(n_i)) \exp(i \nu_i Y(n_i)), \quad (2)$$

where $N(G)$ denotes the set of nodes. In the following we concentrate on one node, and on one or two links, and denote point holonomies by $|\mu, \nu\rangle$ and link holonomies by $|k\rangle$, omitting the indices i and j .

In [4] holonomy operators acting on point holonomies are defined as $SU(2)$ operators, in their action on the states traces of $SU(2)$ generators have to be taken. In the following we take the $U(1)$ operators

$$\hat{U}_x = \exp(i X), \quad (3)$$

and \hat{U}_y analogously, for simplicity and for a more natural action on the functions (2). In their action on arbitrary nodes they raise the labels μ and ν in (2) by one. As indicated above, in [4] the point holonomies are introduced as unitary representations of \mathbf{R}_{Bohr} and holonomies as operators shifting the labels of these representations. Later on, it will turn out that only states $|m, n\rangle$ with integer labels $\mu = m$ and $\nu = n$ are of interest. In the solutions of our model only such series of states out of the representations of \mathbf{R}_{Bohr} contain states with m or n or both being equal to zero. With such a reduction also point holonomies lie in $U(1)$. In section 4 this series will be distinguished as possible sets of kinematical states.

As in [4], the states $|k\rangle$ are considered to lie in $U(1)$. The holonomy operator

$$\hat{U}_\ell = \exp \left(i \int_\ell \mathcal{A} \right) \quad (4)$$

multiplies state functions by an element of the fundamental representation of $U(1)$ and as such it raises the label k of the representation of the state function $|k\rangle$ by one.

The densitized triads E^x and E^y are scalar densities, when integrated over some interval I on the z axis, they give rise to flux operators with nontrivial action when there is a node in I . Then the action of the operators $\bar{E}^x = -i\delta/\delta X$ and $\bar{E}^y = -i\delta/\delta Y$ on a node function is

$$\int_I \bar{E}^x |m, n\rangle = m |m, n\rangle, \quad \int_I \bar{E}^y |m, n\rangle = n |m, n\rangle \quad (5)$$

up to a factor containing the square of the Planck length which we set equal to one. These operators, taken over from [4], are denoted by a bar instead of the usual hat, as we will introduce different operators for the same quantities in the next section.

$\mathcal{E}(z)$, as a scalar, acts directly at the point z . Up to the mentioned type of factor the action is

$$\bar{\mathcal{E}}(z) |k\rangle = k |k\rangle, \quad (6)$$

when z lies on a link with label k . The meaning of $\mathcal{E}(z)$ is the geometrical area of a plane of unit coordinate area, transversal to a link, i. e. a cross-section area of the gravitational wave. When at z there is a node and when k_- and k_+ are the labels of the link functions left and right from z and $|\psi\rangle$ is a SNW function containing $|k_+\rangle$ and $|k_-\rangle$ then

$$\bar{\mathcal{E}}(z) |\psi\rangle = \frac{k_+ + k_-}{2} |\psi\rangle. \quad (7)$$

All these triad operators act diagonally on state functions in the SNW basis.

3 Redefinition of basic operators

In the foregoing section we have briefly introduced the triad operators as in [3] and holonomy operators in a simplified form (from $SU(2)$ to $U(1)$) which, nevertheless, is sufficient for acting on $U(1)$ state functions.

A slightly modified construction of operators starts from the fact that the variables to describe quantum states are functions of group elements, namely $U(1)$ holonomies. Let's take a point holonomy of X (the construction of the Y holonomies is analogous)

$$U_x(z) = e^{imX(z)}. \quad (8)$$

For every node, group elements are labeled by a number $X(z)$ on the manifold of $U(1)$ - a circle - with $0 \leq X(z) < 2\pi$. Integers m label irreducible representations, for $m = 1$ we have the fundamental one, let's denote it by

$$g(z) = e^{iX(z)}. \quad (9)$$

$X(z)$ is a local generator on the one-dimensional space manifold. Quantum state functions at nodes are functions on $U(1) \times U(1)$, a basis for the X functions is given by the point holonomies

$$U_x(z) = e^{imX(z)} = g^m. \quad (10)$$

The basic idea for a modified construction of operators on this space of functions is to replace X and E^x by canonical variables in terms of holonomies:

$$X \rightarrow U_x - \mathbf{1}, \quad E^x \rightarrow -i \frac{\delta}{\delta U_x} = -i \frac{\delta X}{\delta U_x} \frac{\delta}{\delta X} = -U_x^{-1} \frac{\delta}{\delta X}, \quad (11)$$

$\mathbf{1}$ is the unit operator. The multiplication operator $\hat{U}_x(z)$ multiplies the state function at z by the holonomy (9), in other words, it raises the label m by one,

$$\hat{U}_x(z) = e^{iX(z)}, \quad \hat{U}_x(z) |m, n\rangle = |m+1, n\rangle. \quad (12)$$

The derivative operator $i\delta/\delta(X)$ does not commute canonically with \hat{U}_x , the commutator is a holonomy. A canonically conjugate operator to \hat{U}_x is taken from (11)

$$\hat{E}^x := -\hat{U}_x^{-1}(z) \delta/\delta X(z), \quad \hat{E}^x(z)|m, n\rangle = -im|m-1, n\rangle, \quad (13)$$

a lowering operator combined with a multiplier by m . The commutator is

$$[\hat{U}_x(z_i), \hat{E}^x(z_j)] = i\delta_{ij}. \quad (14)$$

when z_i and z_j are the coordinates of nodes.

When we formulate an operator corresponding to X in the form

$$\hat{X}(z) := \hat{U}(z) - \mathbf{1}, \quad (15)$$

it is, of course, also canonically conjugate to \hat{E} ,

$$[\hat{X}(z_i), \hat{E}^x(z_j)] = i\delta_{ij}. \quad (16)$$

$U_x - 1$ is a good approximation in first order for small values of $X(z)$, i.e. for weak fields, when we expect quantum theory to approach the classical limit. The classical expression $-U_x^{-1}E^x$ approximates the conventional differential operator $\delta/\delta X$ in zeroth order in the limit of small X .

Alternatively we can define

$$\tilde{E}^x := -\hat{U}_x(z) \delta/\delta X(z), \quad \tilde{E}^x|m, n\rangle = -im|m+1, n\rangle \quad (17)$$

and replace \hat{X} by

$$\tilde{X}(z) := \mathbf{1} - \hat{U}_x^{-1}(z) \quad (18)$$

with the same commutation relation as (16),

$$[\tilde{X}(z_i), \tilde{E}^x(z_j)] = i\delta_{ij}. \quad (19)$$

From the local variables \mathcal{A} and \mathcal{E} we construct operators acting on link holonomies. The holonomy operator (4) raises the label k of a link by 1,

$$\hat{\mathcal{U}}_\ell |k\rangle = |k+1\rangle. \quad (20)$$

The triad operator $\hat{\mathcal{E}}_\ell$ is constructed from

$$\mathcal{E}_\ell = -\mathcal{U}_\ell^{-1} \frac{\delta}{\delta \mathcal{A}(z)}, \quad z \in \ell, \quad (21)$$

or as alternative version $\tilde{\mathcal{E}}$ analogously to (17), with the respective actions on link functions

$$\hat{\mathcal{E}}_\ell |k\rangle = -ik|k-1\rangle, \quad \tilde{\mathcal{E}}_\ell |k\rangle = -ik|k+1\rangle. \quad (22)$$

The operators constructed from the connection component \mathcal{A} are

$$\hat{\mathcal{A}} := \hat{\mathcal{U}} - \mathbf{1} \quad \text{or} \quad \tilde{\mathcal{A}} := \mathbf{1} - \hat{\mathcal{U}}^{-1}, \quad (23)$$

their corresponding classical expressions are good approximations for \mathcal{A} in first order for short links. The commutators with the triad operators are canonical

$$[\hat{\mathcal{A}}_\ell, \hat{\mathcal{E}}_{\ell'}] = [\tilde{\mathcal{A}}_\ell, \tilde{\mathcal{E}}_{\ell'}] = i \delta(\ell, \ell'). \quad (24)$$

$\delta(\ell, \ell')$ is one if ℓ and ℓ' are the same link, otherwise zero. In the continuous limit (21) approaches a mere derivative operator, as for short links \mathcal{U} and \mathcal{U}^{-1} approach identity in zeroth order.

4 The Killing constraint for unidirectional waves

In [2] the condition of unidirectionality of plane gravitational waves was formulated in form of first class constraints to be imposed in addition to the constraints of canonical general relativity. They are derived from the existence of a null Killing field in the direction of wave propagation and have the form

$$K_\pm := XE^x + YE^y \pm \mathcal{E}'. \quad (25)$$

The prime denotes the derivative with respect to z , the expression is of density weight one. The physical meaning is the following: When the spatial metric (1) is supplemented by a time component $g_{tt} = -g_{zz}$ and zero shift vector, then the classical expression

$$K_1 := XE^x + YE^y \quad (26)$$

is the time derivative of \mathcal{E} (see [3]), thus

$$K_\pm = \dot{\mathcal{E}} \pm \mathcal{E}'. \quad (27)$$

$K_+ = 0$ determines thus waves going into the positive z direction at the speed of light (called right-moving waves) and $K_- = 0$ describes left-moving waves.

In [1] we attempted to quantize this constraint by expressing the first part, K_1 , in terms of a commutator of part of the Hamiltonian constraint with the volume operator. Classically $K_1 = 0$ and $\mathcal{E}' = 0$ together distinguish a state without waves at all, a one-dimensional description of the Minkowski vacuum. As solutions of the corresponding quantum constraint equations we found states that are normalizable, but, with the exception of the zero-volume node state $|0, 0\rangle$, they have diverging expectation values of the length $\sqrt{E^x E^y / \mathcal{E}}$ between two nodes and the volume $\sqrt{\mathcal{E} E^x E^y}$ associated to a node. The situation becomes better when the classical constraint is multiplied by some power of the volume and quantized afterwards. For higher powers the convergence of length

expectation values and fluctuations become increasingly better, but this approach contains an element of arbitrariness - which power should one choose? Moreover, the constraints constructed in this way have different density weights, as the volume is the determinant of the spatial metric.

4.1 Node operators and functions

To obtain a real action of the operator \hat{K}_+ , we multiply (25) by i before defining an operator. Then with the choice (13) and (15) for (X, E^x) and analogously for (Y, E^y) the Killing constraint K_+ acquires the form

$$\hat{K}_+(z) := i\hat{X}(z)\hat{E}^x(z) + i\hat{Y}\hat{E}^y(z) + i\mathcal{E}'(z). \quad (28)$$

In this subsection we anticipate eigenfunctions of $\hat{\mathcal{E}}$ (or $\tilde{\mathcal{E}}$) on the links left and right from z , so that $\mathcal{E}'(z)$ is the difference of eigenvalues, simply an imaginary number (because of the i in (22)). A consistent application of the operator $\hat{\mathcal{E}}/\tilde{\mathcal{E}}$ will be the given in subsection (4.2). Preliminarily we define $\mathcal{D} = i\mathcal{E}'$ with real \mathcal{D} . The action on a node function $|m, n\rangle$ is then

$$\hat{K}_+|m, n\rangle = (\mathcal{D} + m + n)|m, n\rangle - m|m - 1, n\rangle - n|m, n - 1\rangle. \quad (29)$$

\hat{K}_+ contains lowering operators, acting from some state $|m, n\rangle$ with positive m and n into the direction of the m and n axes. When an axis is reached, the creation of new states does not continue beyond it, due to the factors m and n , so the solutions have a finite number of nonvanishing coefficients. Here it is essential that $|m, 0\rangle$ and $|0, n\rangle$ are among the solutions, otherwise the solutions would have an infinite number of states $|m, n\rangle$ and diverging geometric expectation values. This justifies the choice of integer labels in physically relevant node states. The resulting equation for the coefficients $a_{m,n}$ of the states $|m, n\rangle$ in an eigenstate

$$|\mathcal{D}\rangle = \sum_{m,n} a_{m,n} |m, n\rangle \quad (30)$$

of

$$\hat{K}_1 := \hat{X}\hat{E}^x + \hat{Y}\hat{E}^y \quad (31)$$

with eigenvalue $-\mathcal{D} \geq 0$ is the following:

$$(\mathcal{D} + m + n) a_{m,n} - (m + 1) a_{m+1,n} - (n + 1) a_{m,n+1} = 0. \quad (32)$$

Consider first nonpositive integer values of \mathcal{D} :

Case 1. Solutions in the first quadrant of the (m, n) plane, $m \geq 0, n \geq 0$:

1. $\mathcal{D} = 0$: Here the only finite solution is $|0, 0\rangle$.
2. $\mathcal{D} = -1$: There are two solutions,

$$\frac{1}{\sqrt{2}}(|0, 0\rangle - |1, 0\rangle) \quad \text{and} \quad \frac{1}{\sqrt{2}}(|0, 0\rangle - |0, 1\rangle). \quad (33)$$

3. $\mathcal{D} = -2$: Three finite solutions,

$$\begin{aligned} \frac{1}{\sqrt{6}}(|0,0\rangle - 2|1,0\rangle + |2,0\rangle), \quad \frac{1}{\sqrt{6}}(|0,0\rangle - 2|0,1\rangle + |0,2\rangle) \\ \text{and} \quad \frac{1}{2}(|0,0\rangle - |1,0\rangle - |0,1\rangle + |1,1\rangle). \end{aligned} \quad (34)$$

Here appears the first nonzero expectation value $\langle\sqrt{mn}\rangle = \frac{1}{4}$, and fluctuation $\Delta(\sqrt{mn}) = \frac{\sqrt{3}}{4}$ of the node contribution $\sqrt{E^x E^y}$ to length and volume.

For larger negative values of \mathcal{D} a pattern of binomial coefficients appears. For unnormalized states, with $a_{0,0} = +1$ by convention, we find:

4. $\mathcal{D} = -3$: A state with $n = 0$ and the coefficients

$$a_{0,0} = 1, \quad a_{1,0} = -3, \quad a_{2,0} = 3, \quad a_{3,0} = -1,$$

and one containing $n = 0$ and $n = 1$ and the coefficients

$$\begin{aligned} a_{0,0} = 1, \quad a_{1,0} = -2, \quad a_{2,0} = 1, \\ a_{0,1} = -1, \quad a_{1,1} = 2, \quad a_{2,1} = -1 \end{aligned}$$

and two further states with m and n exchanged.

5. $\mathcal{D} = -4$: For $n = 0$ the coefficients are

$$a_{0,0} = 1, \quad a_{1,0} = -4, \quad a_{2,0} = 6, \quad a_{3,0} = -4, \quad a_{4,0} = 1.$$

Then there is a state with $n = 0$ and $n = 1$ and

$$\begin{aligned} a_{0,0} = 1, \quad a_{1,0} = -3, \quad a_{2,0} = 3, \quad a_{3,0} = -1, \\ a_{0,1} = -1, \quad a_{1,1} = 3, \quad a_{2,1} = -3, \quad a_{3,1} = 1, \end{aligned}$$

and finally a state with $n = 0, 1$, or 2 :

$$\begin{aligned} a_{0,0} = 1, \quad a_{1,0} = -2, \quad a_{2,0} = 1, \\ a_{0,1} = -2, \quad a_{1,1} = 4, \quad a_{2,1} = -2, \\ a_{0,2} = 1, \quad a_{1,2} = -2, \quad a_{2,2} = 1, \end{aligned}$$

and two further states with $m \leftrightarrow n$.

One can read off that for each negative integer \mathcal{D} there are $-\mathcal{D} + 1$ solutions with $0 \leq m \leq m_{\max}$ and $0 \leq n \leq n_{\max}$, such that $m_{\max} + n_{\max} = -\mathcal{D}$. The general form of the unnormalized coefficients is

$$a_{m,n} = (-1)^{m+n} \binom{m_{\max}}{m} \binom{n_{\max}}{n}. \quad (35)$$

Case 2. $m \leq 0, n \leq 0$: Here the states $|m-1, n\rangle$ and $|m, n-1\rangle$, which would be created by the above version of the operator from a state $|m, n\rangle$, lie farther

away from the axes than $|m, n\rangle$, so this operator would create an infinity of states with a diverging expectation value of \sqrt{mn} . It is the second version, according to (17) and (18) that acts in this case analogously to the first version in case 1. This can be also expected for reasons of symmetry: As the geometry of $|m, n\rangle$ and $|-m, -n\rangle$ is the same up to the orientation of axes, the operator should act on them in some analogous way, according to what was announced as “principle 2” in the introduction.

Here the Killing operator is explicitly (the following equation defines the operator \tilde{K}_1)

$$\tilde{K}_+(z) = \tilde{K}_1(z) + i\mathcal{E}(z)' := i\tilde{X}(x)\tilde{E}^x(z) + i\tilde{Y}(x)\tilde{E}^y(z) + i\mathcal{E}'(z) \quad (36)$$

and its action on a node state is

$$\tilde{K}_+|m, n\rangle = (\mathcal{D} - m - n)|m, n\rangle + m|m+1, n\rangle + n|m, n+1\rangle. \quad (37)$$

The equation for the coefficients is now

$$(\mathcal{D} - m - n)a_{m,n} + (m-1)a_{m-1,n} + (n-1)a_{m,n-1} = 0. \quad (38)$$

In the result for a given \mathcal{D} we obtain the same type of function as in the foregoing case with the same coefficients $a_{-m,-n} = a_{m,n}$ as the corresponding coefficients for positive m and n , explicitly

$$a_{m,n} = (-1)^{m+n} \begin{pmatrix} -m_{\min} \\ -m \end{pmatrix} \begin{pmatrix} -n_{\min} \\ -n \end{pmatrix}. \quad (39)$$

Case 3. $m \geq 0, n \leq 0$: To obtain an action of the Killing constraint “towards the axes”, XE^x is promoted to an operator according to (13) and (15) and YE^y according to (17) and (18). In this way we obtain again solutions with a finite number of nonzero coefficients. The equation for the coefficients $a_{m,n}$ is the following

$$(\mathcal{D} + m - n)a_{m,n} - (m+1)a_{m+1,n} + (n-1)a_{m,n-1} = 0, \quad (40)$$

their general form is

$$a_{m,n} = (-1)^{m-n} \begin{pmatrix} m_{\max} \\ m \end{pmatrix} \begin{pmatrix} -n_{\min} \\ -n \end{pmatrix}. \quad (41)$$

Now for each $\mathcal{D} < 0$ the location of nonzero coefficients in the fourth quadrant of the (m, n) plane is bounded by the relation $m_{\max} - n_{\min} = -\mathcal{D}$.

Case 4. $m \leq 0, n \geq 0$: This case is analogous to the foregoing one with the roles of m and n exchanged and the solution lying in the second quadrant. In all four cases the coefficients can be normalized according to

$$\bar{a}_{m,n} = (-1)^{|m|+|n|} \left[\begin{pmatrix} 2|m_{\text{m}}| \\ |m_{\text{m}}| \end{pmatrix} \begin{pmatrix} 2|n_{\text{m}}| \\ |n_{\text{m}}| \end{pmatrix} \right]^{-\frac{1}{2}} \begin{pmatrix} |m_{\text{m}}| \\ |m| \end{pmatrix} \begin{pmatrix} |n_{\text{m}}| \\ |n| \end{pmatrix}, \quad (42)$$

where m_m and n_m mean the m or n with the maximal absolute value.

In dependence on the sign of point holonomy labels the X operator, as applied in the above four cases, can be written in the unified form

$$\hat{\hat{X}} = \text{sign}(m) \left(\hat{U}_x^{\text{sign}(m)} - \mathbf{1} \right), \quad (43)$$

acting as \hat{X} or \tilde{X} , according to the sign of m . In the next subsection we will introduce in the same way two versions for link operators in dependence on the link label k , so we may summarize the unified definitions. We write A for X , Y , \mathcal{A} and U_A for the corresponding holonomies, α for the labels m , n , k of a state function, E^A for the conjugate momenta. Then in general the following operators may be defined

$$A \rightarrow \text{sign}(\alpha) \left(U_A^{\text{sign}(\alpha)} - \mathbf{1} \right), \quad E^A \rightarrow -i U_A^{-\text{sign}(\alpha)} \frac{\delta}{\delta A}. \quad (44)$$

So far we have presented four independent solutions to the right-moving unidirectionality constraint, one in each quadrant. At this stage, we can seemingly either restrict ourselves to solutions in one quadrant with one version of the operator, or take together two or all four kinds of solutions. Whether or not one of the latter versions is necessary, depends in the end on the Hamiltonian constraint operator, which determines the dynamics. However, at the kinematical level we did not yet consider the case $\mathcal{D} > 0$, and a discussion of this also involves at least two solutions in two different (m, n) quadrants.

In all four cases considered above the eigenvalues of \hat{K}_1/\tilde{K}_1 are positive. As already mentioned, in classical terms K_1 represents the time derivative $\dot{\mathcal{E}}$, and for right-moving waves, where $\dot{\mathcal{E}} = -\mathcal{E}'$, we have so far obtained only solutions with $\dot{\mathcal{E}} \geq 0$ and $\mathcal{E}' \leq 0$ at every node. When $\mathcal{E}' > 0$ and the wave is going to the right, $\dot{\mathcal{E}}$ must necessarily be negative.

Technically this can be achieved by choosing for the case $\mathcal{E}' > 0$ the quadrant $m \leq 0$, $n \leq 0$ and replace \mathcal{D} by $-\mathcal{D}$ in eq. (38), whereas the first quadrant remains reserved to $\mathcal{E}' < 0$. In this way \mathcal{E}' is changed to $-\mathcal{E}'$, so that the right-moving constraint becomes

$$K_+ = X E^x + Y E^y - \mathcal{E}', \quad (45)$$

which looks formally like K_- in the original definition (25), but as now \mathcal{E} goes to $-\mathcal{E}$, the meaning of K_1 is now $-\dot{\mathcal{E}}$, and $\dot{\mathcal{E}} = -\mathcal{E}'$ again, with $\dot{\mathcal{E}} < 0$ and $\mathcal{E}' > 0$. Rephrasing it in a different way, for $\mathcal{E}' > 0$ we have constructed a solution with opposite orientation, moving to the left and backwards in time, and reinterpret it as right-moving forward in time.

Effectively we redefine K_+ to be given by (25) in the quadrant $m > 0$, $n > 0$ for $\mathcal{E}' < 0$, and by (45) in the quadrant $m < 0$, $n < 0$ for $\mathcal{E}' > 0$. This approach makes use of two quadrants of the (m, n) plane, but one could also find more extended definitions involving all four quadrants. The occurrence of different

signs of triad components, leading to sectors of the theory with different spatial triad orientations, is common in LQG and loop quantum cosmology, see, for example [6, 7].

4.2 Link operators and functions

So far $\mathcal{D} = i\mathcal{E}'$ has been considered simply as an integer number in order to match the integers m and n . But, to be consistent with the foregoing, we must also replace the classical canonical pair $(\mathcal{A}, \mathcal{E})$ by a pair of canonically commuting operators, according to (22) and (23), in dependence of the sign of the link label k .

A single link holonomy $|k\rangle$ is not an eigenstate of $\hat{\mathcal{E}}$ or $\tilde{\mathcal{E}}$. As $\hat{\mathcal{E}}$ is basically a lowering operator, to be applied for $k > 0$, and $\tilde{\mathcal{E}}$ is a raising operator for $k < 0$, eigenstates are in both cases of the type of coherent states in the form of

$$|\kappa\rangle \sim \sum_{k=0}^{\infty} \frac{\kappa^k}{k!} |k\rangle \quad \text{with} \quad \hat{\mathcal{E}} |\kappa\rangle = -i\kappa |\kappa - 1\rangle \quad (46)$$

for $\kappa > 0$ and

$$|\kappa\rangle \sim \sum_{k=0}^{-\infty} \frac{\kappa^{-k}}{(-k)!} |k\rangle \quad \text{with} \quad \tilde{\mathcal{E}} |\kappa\rangle = -i\kappa |\kappa + 1\rangle \quad (47)$$

for $\kappa < 0$. Then the operator \hat{K}_+ in the version (28) ($\bar{\mathcal{E}}$ denotes the \mathcal{E} operator for both $\kappa > 0$ and $\kappa < 0$)

$$\hat{K}_+ = \hat{K}_1 + i\bar{\mathcal{E}}_+ - i\bar{\mathcal{E}}_- \quad (48)$$

with $\bar{\mathcal{E}}_{\pm}$ acting on the links right and left from the considered node. On a state $|\psi\rangle$ containing an eigenstate of \hat{K}_1 with eigenvalue \mathcal{D} (30), as well as eigenstates $|\kappa_{\pm}\rangle$ of $\bar{\mathcal{E}}_+$ and $\bar{\mathcal{E}}_-$,

$$|\psi\rangle = \dots |\kappa_- \rangle \otimes |\mathcal{D}\rangle \otimes |\kappa_+ \rangle \dots, \quad (49)$$

\hat{K}_+ acts in the way

$$\hat{K}_+ |\psi\rangle = (\mathcal{D} + \kappa_+ - \kappa_-) |\psi\rangle \quad (50)$$

and for $\mathcal{D} = \kappa_- - \kappa_+$ we have a solution of the constraint.

Eigenstates of $\hat{\mathcal{E}}$ have the following normalization

$$\langle \kappa | \kappa \rangle = \sum_{k=0}^{\infty} \frac{\kappa^{2k}}{(k!)^2} = I_0(2\kappa). \quad (51)$$

$I_n(x) = (-i)^n J_n(ix)$ are modified Bessel functions. So the normalized eigenfunctions are

$$|\kappa\rangle = \frac{1}{\sqrt{I_0(2\kappa)}} \sum_{k=0}^{\infty} \frac{\kappa^k}{k!} |k\rangle. \quad (52)$$

With this normalization the expectation value of a positive k in an eigenstate becomes

$$\langle k \rangle = \kappa \frac{I_1(2\kappa)}{I_0(2\kappa)}, \quad (53)$$

with the fluctuation

$$\Delta k = \kappa \sqrt{1 - \left(\frac{I_1(2\kappa)}{I_0(2\kappa)} \right)^2}. \quad (54)$$

For growing κ the area expectation value $\langle k \rangle$ quickly approaches κ , whereas the area fluctuation Δk grows only slowly (for example $\Delta k \approx 22$ for $\kappa = 1000$.) This is in accordance with the fact that for weak gravitational waves transversal area variations are small.

5 Conclusion

In the present approach we have found well-behaved kinematical solutions to the unidirectionality constraint for plane gravitational waves. For complete solutions we need two sectors of the theory with different signs of triad variables. These different signs distinguish different orientations of space. For physical reasons one may expect the action of quantum operators on states with only sign differences to act in a very closely related way, as operator pairs like (\hat{X}, \tilde{X}) , (\hat{E}^x, \tilde{E}^x) or (\hat{K}_1, \tilde{K}_1) (defined in (31) and (36)) do in their respective domains.

Even if from the mathematical point of view the approach with canonically commuting operators may appear less natural, physically it yields substantially better results than the previous one in [1], where more common methods were used. After all, with the aid of step and sign functions and \hat{K}_1 and \tilde{K}_1 it is possible to formulate the right-moving constraint operator constructed in this paper in a closed form, when the link functions are coherent states:

$$\begin{aligned} \widehat{\hat{K}}_+ &:= \Theta(m)\Theta(n)\hat{K}_1 + \Theta(-m)\Theta(-n)\tilde{K}_1 + \\ &\text{sign}(\kappa_+ - \kappa_-)[\Theta(k_+)\hat{\mathcal{E}}_+ + \Theta(-k_+)\tilde{\mathcal{E}}_+ - \Theta(k_-)\hat{\mathcal{E}}_- - \Theta(-k_-)\tilde{\mathcal{E}}_-]. \end{aligned} \quad (55)$$

Whether or not dynamical solutions can be of the considered type, or whether all four quadrants of (m, n) are needed for a consistent dynamics of the model, is expected to be determined by the action of the Hamiltonian constraint on the states found in this paper. This problem will be the subject of future work.

It also turned out that at the kinematical level the Minkowski vacuum cannot be modeled by solutions of both the quantum constraints corresponding to the classical constraints $K_1 = 0$ and $\mathcal{E}' = 0$. Assuming globally $\mathcal{E} = \text{const.}$, we are left with the zero volume and zero length state $|0, 0\rangle$ at every node, effectively the same as a state without nodes at all.

Now, as strictly constant cross section area along the z axis is impossible, there must be small, but nonvanishing spatial fluctuations in \mathcal{E} . In the sequel,

area fluctuations lead to volume and length fluctuations, as it follows from the calculations in this paper. This indicates that fluctuations have the form of small left- or right-moving waves, there are no static fluctuations at nodes, while \mathcal{E}' would be zero. Again, what these fluctuations are like in a realistic dynamical model is a matter of the Hamiltonian constraint. In a dynamical Minkowski space solution we can expect a balanced mixture of right- and left-moving fluctuations everywhere along the z axis.

References

- [1] J. Adelman, F. Hinterleitner, S. Major, Quantum volume and length fluctuations in a midi-superspace model of Minkowski space, *Class. Quantum Grav.* **32**, 5 (2015) 055009 (arXiv:1401.0327)
- [2] F. Hinterleitner, S. Major, Towards Loop Quantization of Plane Gravitational Waves, *Class. Quantum Grav.* **29** (2012) 065019, (arXiv:1106.1448)
- [3] K. Banerjee, G. Date, Loop quantization of polarized Gowdy model on T^3 : classical theory, *Class. Quantum Grav.* **25** (2008) 105014 (arXiv:0712.0683)
- [4] K. Banerjee, G. Date, Loop quantization of polarized Gowdy model on T^3 : kinematical states and constraint operators, *Class. Quantum Grav.* **25** (2008) 145004 (arXiv:0712.06887)
- [5] D. Martín de Blas, J. Olmeda, T. Pawłowski, Loop quantization of the Gowdy model with local rotational symmetry, (arXiv:1509.09197)
- [6] M. Bojowald, *Loop Quantum Cosmology*, Springer, New York, 2011
- [7] C. Kiefer, C. Schell, Interpretation of the triad orientations in loop quantum gravity, *Class. Quantum Grav.* **30** (2013), 035008, (arXiv:1210.0418)